# Readers' Forum

*Brief discussions of previous investigations in the aerospace sciences and technical comments on papers published in the* AIAA Journal *are presented in this special department. Entries must be restricted to a maximum of 1000 words, or the equivalent of one* Journal *page including formulas and figures. A discussion will be published as quickly as possible after receipt of the manuscript. Neither the AIAA nor its editors are responsible for the opinions expressed by the correspondents. Authors will be invited to reply promptly.*

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## Comment on "Linear and Nonlinear Analysis of a Nonconservative Frame of Divergence Instability"

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RECENTLY Kounadis and Avraam[1] presented an analysis of the divergence instability of nonconservatively loaded frames. The purpose for their work, as expressed in their introduction, needs some clarification. Furthermore, the nonlinear static analysis contains an error that, although it may not seriously affect the results and conclusions of Ref. 1, is worth noting because it could affect the results for more complicated systems.

There are three questions raised in the introduction of Ref. 1, that, according to the authors, need to be addressed in further research. The first question concerns "the failure of static methods of analysis of [nonconservatively loaded] systems due to the fact that nonlinear terms are not taken into account." It is well known, however, that the "failure" of static methods often spoken of in older papers is not a failure at all. A dynamic analysis is required to predict dynamic instability (flutter), and a static analysis is sufficient to predict static instability (divergence).[2] Addition of nonlinear terms is necessary in either case when geometric nonlinearities are expected to play an important role in the calculation of static equilibrium. The presence of nonlinear terms, however, has nothing to do with changing the fundamental distinction between flutter and divergence. To be consistent with the state-of-the-art, linear dynamic analysis should be done with the dynamic equations linearized about the equilibrium state established from the nonlinear static equations.[3] The nonlinear portion of the analysis can be accomplished in a straightforward way for the problem treated in Ref. 1 for small strain but without usual small-angle restrictions on rotations. In order to illustrate this point and simultaneously deal with the error in Ref. 1, a brief derivation is helpful.

The general nonlinear Euler-Bernoulli beam kinematic analysis presented in Ref. 4 is simplified herein to the simpler planar case of Ref. 1. The direction cosines of the beam axis during deformation can be related to fixed coordinate axes in terms of one angle, say $\beta$:

$$i=\cos\beta I+\sin\beta J, \qquad j=-\sin\beta I+\cos\beta J \tag{1}$$

where $i$ is the unit vector normal to the beam cross section during deformation and $j$ is the unit vector normal to $i$ and in the plane of deformation. Prior to deformation, unit vectors $i$ and $j$ coincide with $I$ and $J$, unit vectors along fixed coordinate axes. The position vector from the beam root to a generic point $P$ on the elastic axis is, in the notation of Ref. 1,

$$r=(x+\xi)I+wJ \tag{2}$$

The unit vector tangent to the elastic axis at $P$ is

$$\frac{dr}{ds}=\frac{d}{ds}(x+\xi)I+\frac{dw}{ds}J \tag{3}$$

where $s$ is the distance along the deformed beam elastic axis.

If shear deformation is ignored, $\beta$ can be eliminated by requiring $i$ to be tangent to the deformed beam elastic axis:

$$i=dr/ds \tag{4}$$

so that

$$\sin\beta=dw/ds, \quad \cos\beta=d(x+\xi)/ds \tag{5}$$

Because $\sin^2\beta+\cos^2\beta=1$, Eqs. (5) yield the differential relationship

$$(dx+d\xi)^2+dw^2=ds^2 \tag{6}$$

or, in terms of derivatives with respect to $x$,

$$s'=[(1+\xi')^2+w'^2]^{1/2} \tag{7}$$

where $(\ )'=d/dx(\ )$. The value of $s'$ is near unity for small strain. In fact $s'-1$ is quite appealing as a force strain measure because it is linear in the elongation. Now, unit vectors $i$ and $j$ as given in Eq. (1) can be expressed in terms of $\xi'$ and $w'$ only where

$$\sin\beta=w'/s', \quad \cos\beta=(1+\xi')/s'=(s'^2-w'^2)^{1/2}/s' \tag{8}$$

A suitable moment strain† is given by Reissner[5] as $\kappa=\beta'$. Use of Eqs. (8) to express $\beta'$ in terms of $\xi$ and $w$ and their derivatives yields

$$\kappa=\beta'=\frac{(w'/s')'s'}{1+\xi'}=\frac{(w'/s')'s'}{(s'^2-w'^2)^{1/2}} \tag{9}$$

A considerable simplification results from ignoring the axial strain compared to unity in the direction cosines and moment strain. This can be accomplished by setting $s'=1$ in Eqs. (8) and (9) so that

$$\sin\beta\cong w'$$

$$\cos\beta\cong(1-w'^2)^{1/2}$$

$$\kappa\cong w''/(1-w'^2)^{1/2} \tag{10}$$

The singularity in the kinematics is now clearly revealed in the expression for $\kappa$ at $w'=1$ (or $\beta=90$ deg). Other than this obvious restriction, however, Eqs. (10) are valid for small strains and orientation angles up to $\beta=90$ deg. It is quite interesting, as an aside, to compare the expression for $\kappa$ [Eqs.

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†Note that the moment strain used here, although it has dimensions of curvature, is not curvature.

(10)] with another candidate moment strain

$$\bar{\kappa}=w''/(1+w'^2)^{3/2} \tag{11}$$

that, at first, may seem more likely to be correct. The difference in the two geometrically is that while $\bar{\kappa}$ is the curvature of the curve $w$ vs $x$ (which has no physical significance for this problem), $\kappa$ is approximately (exactly for $s'=1$) the physical curvature of the beam elastic axis which is nearly inextensional for small strain. Thus, the bending moment is proportional to $\kappa$, and the strain energy can now be written as

$$U=\frac{1}{2}\int_0^\ell [EA(s'-1)^2+EI\kappa^2]\mathrm{d}x \tag{12}$$

The virtual work, including terms associated with a follower force $-Pi(\ell)$, is then given by

$$\delta U+P[\delta\xi(\ell)\cos\beta(\ell)+\delta w(\ell)\sin\beta(\ell)]=0 \tag{13}$$

with $U$ given by Eq. (12) and $s'$, $\kappa$, $\sin\beta$, and $\cos\beta$ expressed from Eqs. (7) and (10). Boundary conditions follow naturally from Eq. (13) that will not and cannot contain quantities like $\sin w'$ as do the boundary conditions depicted in Ref. 1. Thus, it is apparent that the boundary conditions given in Ref. 1 are incorrect, strictly speaking, although numerical results may not be appreciably different when the present equations are used. It is also now apparent that an analysis similar to that of Ref. 1 can be formulated that is restricted to small strains but has no restriction on rotations other than the singularity at $\beta=90$ deg. A dynamic analysis linearized about the equilibrium condition thus obtained would be consistent with the state of the art in elastic stability theory and may lead to improved understanding of the phenomena discussed in Ref. 1.

## References

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# Reply by Authors to D. H. Hodges

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THE authors would like to express their thanks to Dr. Hodges for his Comment and to point out the following:

1) An exact investigation of the stability of elastic systems subjected to follower forces entails considerable computational difficulties. This is evident if the mathematical formulation of such a problem is based on small strains but large rotations. The expressions for the axial and moment strain, strain energy, $\sin\beta$, and $\cos\beta$ given by Dr. Hodges are well known.[1-3] The authors, in order to overcome these mathematical difficulties, have chosen to employ the theory of intermediate deformation[4] (small strains but moderately large rotations) which gives excellent results within a wide range.

2) Clearly, the difference between $\sin\beta$ (which theoretically should be present in the boundary conditions) and $\sin w'$ (where $w'=\tan\beta$) due to the small rotations involved is negligible. However, the numerical results presented in the work under discussion are associated with Eqs. (20), which are based on the following aprroximations: $\sin\beta\simeq w'$, $\cos\beta\simeq 1$ and $u=-w''$. These approximations are consistent with the aforementioned theory. A detailed derivation of all nonlinear stability equations of this work is presented in Ref. 5, where use of these approximations is made. The authors are presently employing a more inclusive theory to solve the same problem.

3) An extension of the foregoing work in which several aspects of this problem are clarified is available in Ref. 6.

4) From this work and that of Ref. 6 it is indicated that the nonlinear terms have the following result: the range of values of parameters for which according to linear stability analysis (either dynamic or static) flutter occurs might be appreciably reduced, if a nonlinear static analysis is employed.

## References

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[5]Avraam, T. P., Pantis, M. A., and Kounadis, A. N., "Snap-Through Buckling of a Simple Frame with a Tangential Load," *Acta Mechanica*, Vol. 36, 1980, pp. 119-127.

[6]Kounadis, A. N., "Postbuckling Analysis of Divergence and Flutter Instability Structures under Tangential Loads," *Proceedings of the 2nd Symposium on Stability in the Mechanics of Continua*, Aug. 31-Sept. 4, 1981, Nümbrecht, W. Germany (to be published).

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# Comment on "Potential of Transformation Methods in Optimal Design"

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IN a recent Note,[1] Belegundu and Arora presented an approach for computing the derivatives of a penalty function directly without calculating derivatives of individual constraints. The penalty function acts as an equivalent constraint replacing all other constraints. Their approach permits computation of the first derivatives of the penalty function with one forward and backward substitution (FBS) of the stiffness matrix equation. Newton's method is a powerful technique, for it finds a minimum of a quadratic

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